RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

M.A./M.Sc. SECOND SEMESTER EXAMINATION, MAY 2019

FIRST YEAR [BATCH 2018-20]

MATHEMATICS

Paper : VIII (Real Analysis II)

Date : 17/05/2019

Time : 11 am – 1 pm

Full Marks : 50

[Answer as many questions as you possibly can. Maximum you can score is 50.]

1. Let f denote a function on the interval $[-\pi, \pi]$ such that

   i) f is continuous on $[-\pi, \pi]$

   ii) $f(-\pi) = f(\pi)$

   iii) $f^{\prime}$ is piecewise continuous on the interval $(-\pi, \pi)$.

   If a_n, b_n are the Fourier coefficients corresponding to f, then show that the series

   $\sum_{n=1}^{\infty}\left(a_n^{\,2} + b_n^{\,2}\right)^{\frac{1}{2}}$ converges . [10]

2. Show that the Dirichlet kernel $D_N(u) = \frac{1}{2} + \sum_{n=1}^{N} \cos nu$ can be written as

$$D_N(u) = \frac{\sin\left(\frac{u}{2} + Nu\right)}{2\sin\frac{u}{2}} \qquad (u \neq 2p\pi, p \in \mathbb{Z})$$ [4]

3. Let f be defined on a neighbourhood of 0. If f(0+) exists and for some $\varepsilon > 0$, the Lebesgue integral $\int_0^{\varepsilon} \frac{f(x) - f(0+)}{x}\, dx$ exists ,then show that $\lim_{m\to\infty} \frac{2}{\pi}\int_0^{\varepsilon} f(x)\frac{\sin mx}{x}\, dx = f(0+)$. [4]

4. Let $f(x) = \|x\|^4, x \in \mathbb{R}^n$ .Find the first order partial derivatives of f and the directional derivatives of f along $u = (1,1,......,1,1) \in \mathbb{R}^n$. [4]

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined as $f(x, y) = (\sin x + \cos y, \cos x \sin y, \sin x \sin y)$.
   Find the Jacobian matrix $Df(x, y)$. [5]

6. State and prove the Implicit function theorem for a function $f : \mathbb{R}^{n+k} \to \mathbb{R}^n$. [10]

7. Find the maximum value of $\left(x_1 x_2 .... x_n\right)^2$ where $x = \left(x_1, x_2, ..., x_n\right) \in \mathbb{R}^n$ lie on the unit sphere centered at the origin. [6]

8. Define sequences $\{f_n\}$ and $\{g_n\}$ as follows:

$$f_n(x) = x\left(1 + \frac{1}{n}\right),\ x \in \mathbb{R}, n \in \mathbb{N}$$

$$g_n(x)=\begin{cases}\frac{1}{n}, \text{if } x\in\mathbb{Q}^c\cup\{0\}\\ \\ b+\frac{1}{n}, \text{ if } x=\frac{a}{b}, b>0, a\in\mathbb{Z}\setminus\{0\}\end{cases}$$

Let $h_n(x)=f_n(x)g_n(x)$ .If I be a bounded interval , check the uniform convergence of $\{f_n\},\{g_n\}$ and $\{h_n\}$ on I . [7]

9. Find the radius of convergence of the power series

$$1+\frac{a.b}{1.c}x+\frac{a(a+1)b(b+1)}{(1.2)c(c+1)}x^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{(1.2.3)c(c+1)(c+2)}x^3+\text{..........},$$

where . in the denominator denotes product. [4]

10. Prove that

a) $B(m,m).B\left(m+\frac{1}{2},m+\frac{1}{2}\right)=\frac{\pi}{m}2^{1-4m}, m>0$ [3]

b) $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\text{.....}\Gamma\left(\frac{8}{9}\right)=\frac{16\pi^4}{3}$ [3]

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